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NOTE ON SERIES OF ANALYTIC FUNCTIONS*

BY CESARE ARZELA

I.

IN the "Note on the functions defined by infinite series whose terms are analytic functions of a complex variable, etc.,†" Professor Osgood has demonstrated the following theorem, the generality and importance of which are obvious.

"*If the infinite series*

$$f_1(z) + f_2(z) + \cdots \tag{1}$$

whose terms are all functions of z*, single valued and analytic throughout a region* T *of the complex* z*-plane, converges for all values of* z *pertaining to a set of points which is everywhere dense throughout* T *(and which, in particular, may be enumerable); and if, furthermore, the relation*

$$|f_1(z) + \cdots + f_n(z)| \leqq G$$

holds for all points z *of this set (and hence of* T*), and for all values of* n*,* G *being a positive constant, then the series converges for all values of* z *in* T *and the function* $F(z)$ *defined by the series:*

$$F(z) = f_1(z) + f_2(z) + \cdots$$

is analytic throughout T."

The simplicity of the following demonstration, which depends on certain propositions already obtained by the author, may justify its publication.

II.

The propositions in question are the following:

If there be in a region T of the x, y plane an infinite set of functions

$$(a) \qquad u_1(x, y),\ u_2(x, y),\ \cdots,\ u_s(x, y),\ \cdots,$$

each continuous and included between the fixed numbers l and L then:

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* The editors acknowledge their indebtedness to Dr. C. N. Haskins, who has kindly performed the task of translating this article from the Italian manuscript, and of adding in §VII a translation of an extract from one of the author's earlier papers.

† Osgood, ANNALS OF MATHEMATICS, ser. 2, vol. 3 (1901–02), p. 26.

1. The necessary and sufficient condition that there exist at least one continuous limiting function for this set is that the set contain an infinite number of functions which are *equably continuous.*

2. A set of functions taken from the set a is said to be *equably continuous* in T if, given an arbitrarily small positive number σ, a positive number δ can always be found such that in every part of T whose maximum chord does not exceed δ the oscillation of each of the functions in question is less than σ.

3. A *sufficient* condition for the equable continuity of such a set of functions is that the two difference quotients

$$\frac{u(x_1, y) - u(x_2, y)}{x_1 - x_2}, \qquad \frac{u(x, y_1) - u(x, y_2)}{y_1 - y_2}$$

remain between two finite numbers l and L, for all functions $u(x, y)$ of the set, and for all points (x_1, y), (x_2, y), (x, y_1), (z, y_2) possible in T.

4. If the partial derivatives

$$\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$$

exist, it is sufficient that in T, including the contour, they lie between fixed numbers.

These propositions are demonstrated by methods similar to those used by the author in an earlier paper for the corresponding propositions in the theory of functions of a single variable.*

III.

Let us recall further from the theory of analytic functions of a single variable the propositions :—

1. Given an ordered set,

$$\phi_1(z), \quad \phi_2(z), \cdots, \quad \phi_s(z), \cdots$$

of single valued analytic functions uniformly convergent in T to a function $f(z)$, then this function is in T a single valued analytic function.

2. The series of derivatives

$$\phi_1'(z), \quad \phi_2'(z), \cdots, \quad \phi_s'(z), \cdots$$

converges uniformly to $f'(z)$.

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* See §VII where those portions of this paper with which we are here concerned are reproduced.

IV.

The above being assumed, let us consider a succession of functions single valued and analytic in T:

$$\phi_1(z), \quad \phi_2(z), \cdots, \quad \phi_s(z), \cdots, \tag{1}$$

for each of which there exists at least one point z (which may vary from function to function) in which

$$|\phi_s(z)| < M, \tag{2}$$

M being a fixed number (the same for every z in T) and

$$|\phi_s'(z)| < L, \tag{3}$$

L being likewise a fixed number. Then there exists at least one single valued analytic function $S(z)$ to which *one* succession of functions taken from (1) converges uniformly.

If we put $z = x + yi$, $\phi_s(z) = u_s(x, y) + iv_s(x, y)$ then from (2) follows that, in some point (x, y), we have

$$\begin{aligned} |u_s(x, y)| &< M, \\ |v_s(x, y)| &< M, \end{aligned} \tag{4}$$

whatever be s. The point (x, y), however, may vary with s. From

$$\phi_s'(z) = \frac{\partial u_s}{\partial x} + i\frac{\partial v_s}{\partial x} = \frac{\partial v_s}{\partial y} - i\frac{\partial u_s}{\partial y}$$

follows that the condition (3) is equivalent to the following:

$$\begin{aligned} &\left|\frac{\partial u_s}{\partial x}\right| < L, \qquad \left|\frac{\partial u_s}{\partial y}\right| < L, \\ &\left|\frac{\partial v_s}{\partial x}\right| < L, \qquad \left|\frac{\partial v_s}{\partial y}\right| < L. \end{aligned} \tag{5}$$

This last condition gives, by means of §II, (4), that the successions

$$u_1(x, y), \quad u_2(x, y), \cdots, \quad u_s(x, y), \cdots, \tag{6}$$

$$v_1(x, y), \quad v_2(x, y), \cdots, \quad v_s(x, y), \cdots \tag{7}$$

are equably continuous.

For, from the identity

$$u(x+h, y+k) - u(x, y) = \frac{u(x+h, y+k) - u(x+h, y)}{k} k$$
$$+ u \frac{(x+h, y) - u(x, y)}{h} h$$

(in which (x, y) may be for each of the functions the point in which the relations (4) are true, and $(x+h, y+k)$ any point whatever in T), and by means of (5), we see that all the $u_s(x, y)$ remain always in absolute value less than a fixed number*

$$M + pLd,$$

where d is the longest chord which can be drawn in T, and p is a fixed integer. The same holds for the $v_s(x, y)$. The succession of functions (6) satisfies the conditions of proposition (2) §II. There is, therefore, for this succession at least one continuous limiting function $u(x, y)$.

The particular succession of functions converging uniformly to $u(x, y)$ may for example be chosen by taking from (6) those functions of smallest index, s, satisfying the inequalities

$$|u(x, y) - u_{s_1}(x, y)| < \epsilon,$$
$$|u(x, y) - u_{s_2}(x, y)| < \frac{\epsilon}{2},$$

ϵ being a small positive number which may be taken at pleasure.

Under this assumption, the succession

$$(8) \qquad u_{s_1}(x, y), \quad u_{s_2}(x, y), \cdots$$

converges uniformly to $u(x, y)$ throughout T. The succession of associated functions

$$(9) \qquad v_{s_1}(x, y), \quad v_{s_2}(x, y), \cdots$$

will also admit continuous limiting functions. We cannot say at once that it will admit but one. Let, however, $v'(x, y)$, $v''(x, y)$ be two such functions. Then we shall find that their difference is a constant. For let $v'(x, y)$ be the limiting function to which the succession

$$(9') \qquad v_{s_{n_1}}(x, y), \quad v_{s_{n_2}}(x, y), \cdots$$

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* By means of lines parallel to the axes and wholly contained in T, in number fewer than a fixed number p, we may always join the point (x, y) to the point $(x+h, y+k)$.

taken from (9) converges uniformly, and similarly let $v''(x, y)$ be the function to which the succession

$$(9'') \qquad v_{s_{m_1}}(x, y), \quad v_{s_{m_2}}(x, y), \cdots$$

converges uniformly. Then if we consider the succession of functions associated with (9′) and (9″), *i. e.*

$$u_{s_{n_1}}(x, y), \quad u_{s_{n_2}}(x, y), \cdots$$

and

$$u_{s_{m_1}}(x, y), \quad u_{s_{m_2}}(x, y), \cdots,$$

these two, forming a part of (8), converge to a single function $u(x, y)$. But the two successions of analytic functions

$$u_{s_{n_1}}(x, y) + iv_{s_{n_1}}(x, y), \quad u_{s_{n_2}}(x, y) + iv_{s_{n_2}}(x, y) \cdots$$

and

$$u_{s_{m_1}}(x, y) + iv_{s_{m_1}}(x, y), \quad u_{s_{m_2}}(x, y) + iv_{s_{m_2}}(x, y) \cdots$$

converge uniformly to the functions

$$u(x, y) + iv'(x, y), \quad u(x, y) + iv''(x, y)$$

respectively, which will be, therefore, single valued and analytic in T. From this it follows that $v'(x, y)$ and $v''(x, y)$, as asserted, differ at most by a constant. The proposition is proved.

V.

Consider now Osgood's hypothesis.

1. *Let* $f_1(z), f_2(z), \ldots$ *be analytic functions in* T, *and, letting*

$$s_n(z) = f_1(z) + f_2(z) + \cdots + f_n(z),$$

suppose we have for every n, *and for every* z *in* T,

$$|s_n(z)| < L,$$

where L is fixed.

For a moment let us suppose T simply connected. Then consider the series

$$\int_{z_0}^{z} f_1(z)\,dz + \int_{z_0}^{z} f_2(z)\,dz + \cdots,$$

and let

$$S_n(z) = \int_{z_0}^{z} f_1(z)\,dz + \int_{z_0}^{z} f_2(z)\,dz + \cdots + \int_{z_0}^{z} f_n(z)\,dz,$$

z_0 being an arbitrary fixed point and z any variable point whatever.

The path of integration can always be considered as of length inferior to a given number H. It follows that

$$|S_n(z)| < HL,$$

for every n and for every z. But further

$$S'_n(z) = f_1(z) + f_2(z) + \cdots + f_n(z) = s_n(z),$$

and consequently, by hypothesis, for every n and for every z,

$$|S'_n(z)| < L.$$

Consider now the succession

$$S_1(z),\ S_2(z),\ \cdots,\ S_n(z),\ \cdots.$$

There must exist one or more single valued analytic functions to each one of which converge uniformly successions of functions obtained from the given one. If $\Sigma(z)$ is one of these limiting functions, for example the one to which the succession

$$S_{n_1}(z),\quad S_{n_2}(z),\ \cdots$$

converges, then the succession of the derivatives,

$$S'_{n_1}(z),\quad S'_{n_2}(z),\ \cdots,$$

will admit for limiting function the derivative $\Sigma'(z)$ which will be, moreover, an analytic function.

2. Let us now adjoin the other hypothesis, viz., *that the given series*

$$(a) \qquad f_1(z) + f_2(z) \cdots$$

shall be convergent in the points of a set everywhere dense in T. From this follows that any succession taken from the above, as

$$S'_1(z),\quad S'_2(z),\ \cdots,$$

converges always to a single limiting function.

For if $\Sigma'_1(z)$, $\Sigma'_2(z)$ are two such limiting functions, viz. those of the partial successions

$$S'_{m_1}(z),\quad S'_{m_2}(z),\ \cdots,\quad S'_{m_s}(z),\ \cdots,$$

$$S'_{n_1}(z),\quad S'_{n_2}(z),\ \cdots,\quad S'_{n_s}(z),\ \cdots,$$

respectively, then in all points z of the set in question they have the same limit, viz., that of the proposed series a; therefore, in all points of the everywhere dense set of points the two functions $\Sigma_1'(z) = \Sigma_2'(z)$ coincide. They are continuous in those points, so that we have in all points of T

$$\Sigma_1'(z) = \Sigma_2'(z).$$

From this we have

$$\Sigma_1(z) - \Sigma_2(z) = C = \text{constant},$$

and since, in the point z_0, all of the integrals

$$\int_{z_0}^{z} f_1(z)\,dz\,, \quad \int_{z_0}^{z} f_2(z)\,dz, \ldots$$

are zero, it follows that $C = 0$ and $\Sigma_1(z) = \Sigma_2(z)$.

We have, therefore, proved that, under the assumed hypothesis, the functions $S_n(z)$ converge with increasing n uniformly in T to a single valued analytic function

$$S(z) = \int_{z_0}^{z} f_1(z)\,dz + \int_{z_0}^{z} f_2(z)\,dz + \cdots.$$

But then the derivative of this function is also a single valued analytic function

$$S'(z) = f_1(z) + f_2(z) + \cdots,$$

and that was precisely what we wished to prove.

VI.

We have supposed in the demonstration of the last theorem that the region T was simply connected, but this restriction can be easily removed.

The region T in which the series

$$f_1(z) + f_2(z) + \cdots$$

on may be multiply connected. We reduce the problem by making it simply connected by introducing suitable cuts. Let T' be the new region. By the preceding proposition the sum of the series

$$S(z) = f_1(z) + f_2(z) + \cdots$$

will be a single valued analytic function in T'. But we observe that the cuts may be deformed and that we may obtain from T another region T'', also simply connected, such that the original cuts fall wholly in T'', in which the $S(z)$ is by the same proposition a single valued analytic function.

$S(z)$ is therefore a single valued analytic function in those parts of T in which occur the cuts which form the boundary of T'; and therefore is likewise so in T.

VII.*

We shall now prove the propositions of which those of §II are the extension.

1. THEOREM. The necessary and sufficient conditions that a given succession of functions

$$(a) \qquad u_1(x), \quad u_2(x), \cdots,$$

defined in an interval $a \ldots b$, may converge to a limiting function is that, given an arbitrarily small positive number σ there can always be determined a corresponding integer m_σ such that for all values of x in the interval $a \ldots b$, and for all positive integers p,

$$|u_{m_\sigma}(x) - u_{m_\sigma+p}(x)| < \sigma.$$

The proof of this proposition is similar to that of the corresponding proposition for a succession of *numbers*

$$a_1, a_2, \cdots,$$

and may be omitted here.

2. DEFINITION. *A set of functions*

$$u_1(x), \ u_2(x), \cdots, \ u_s(x), \cdots,$$

all lying between two fixed limits l and L, is said to be *equably continuous* if for every arbitrarily chosen positive number, σ, there always exists a positive number, δ, such that, in every interval of length less than δ, all the functions $u_s(x)$ oscillate by less than σ.

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* §VII is a translation of certain parts of the author's article "Sulle funzioni di linee," *Bologna, Accademia delle Scienze, Memorie*, ser. 5, vol. 5 (1894), p. 225.

A more general proposition than 3 is demonstrated in the memoir "Sulle esistenza degli integrali nelle equazioni differenziali ordinarie," *Ibid*, vol. 6 (1896), p. 131; also in the memoir "Sulle serie di funzioni," *Ibid*, vol. 8 (1899), p. 131 and p. 701; and (for the enunciation at least) in the note "Sul secondo teorema della media per gli integrali doppi," *Ibid*, vol. 10 (1902), p. 99.

Concerning propositions 3 and 5 and also Osgood's theorem it may be noted that an analogous but somewhat less general theorem was stated by de la Vallée-Poussin in 1893.

The consideration of the derivatives for the purpose of determining the equable continuity of an infinite set of functions occurs, though under the restrictive hypothesis of the monotonic increase or decrease of all the functions, in the memoir of Ascoli, "Le curve limite di una varietà data di curve," *Atti della R. Accad. dei Lincei, Memorie*, ser. 3, vol. 18 (1882–83), p. 521; see also *R. Istituto Lombardo, Rendiconti*, ser. 2, vol. 21 (1884).

3. **Theorem.** The necessary and sufficient condition that a set of continuous functions admit a continuous limiting function is that there exist among the set an infinite number of functions which are equably continuous.

The condition is *necessary*.

For suppose $v(x)$ be a limiting function for the functions

$$(a) \qquad u_1(x), \quad u_2(x), \cdots$$

of the succession.

Taking σ arbitrarily we can by (1) find the corresponding m_σ. Since $u_{m_\sigma}(x)$ is continuous, we can find a δ such that, in every interval of length less than δ, $u_{m_\sigma}(x)$ will oscillate by less than σ.

It follows from the inequality in (1) that each of the functions

$$u_{m_\sigma+1}(x), \quad u_{m_\sigma+2}(x), \cdots, \quad u_{m_\sigma+p}(x), \cdots$$

oscillates by less than 3σ in every interval of length less than this same δ.

The functions

$$u_1(x), \quad u_2(x), \cdots, \quad u_{m_\sigma-1}(x)$$

preceding $u_{m_\sigma}(x)$ are finite in number, and hence we can find a number δ_0 such that in every interval of length less than δ_0 they oscillate by less than 3σ. Hence in every interval of length less than the smaller of the two numbers δ and δ_0 each of the functions

$$u_1(x), \quad u_2(x), \cdots, \quad u_n(x), \cdots$$

oscillates by less than 3σ. But 3σ may be *any number whatever*, and, consequently, the equable continuity of the functions is established.

The condition is therefore *necessary*. It is also *sufficient*.

For suppose the functions

$$u_1(x), \quad u_2(x), \cdots, \quad u_n(x), \cdots$$

are equably continuous. Then, given the arbitrarily small positive number σ, we can find a positive number δ such that each of the functions $u_n(x)$ can oscillate by more than σ only in intervals of length greater than δ. For since the functions $u_n(x)$ are equably continuous, we can find for every positive σ a δ such that $u_n(x)$ oscillates by less than σ in every interval of length less than δ.

Consider now the absolute value,

$$|u_p(x) - u_q(x)|,$$

of the difference between any two functions of the set. This difference may in some point x be less than σ, in others equal to or greater than σ.

Let us choose now from the given functions a all those functions,

$$(\beta) \qquad u_{s_1}(x), \quad u_{s_2}(x), \cdots,$$

such that the difference,

$$|u_{s_p}(x) - u_{s_q}(x)|,$$

of any two of them is in some point x greater than or equal to σ, but in all other points is less than σ. Then we can show that the succession β contains at most a finite number of functions.

The functions β are by hypothesis equably continuous. Each of them can therefore oscillate by more than $\sigma/4$ only if x traverses an interval of length greater than a certain assignable value δ'.

Hence if x is a point in which

$$|u_{s_p}(x) - u_{s_q}(x)| \geqq \sigma$$

there will be an interval at least equal to δ' in which the two functions differ by a quantity not less than $\sigma/2$.

Suppose now the functions β were infinite in number. The differences

$$|u_{s_1}(x) - u_{s_2}(x)|, \ |u_{s_1}(x) - u_{s_3}(x)|, \cdots$$

are each, in some interval δ', always greater than or equal to $\sigma/2$; consequently there is a point x_0 which belongs to an infinite number of these intervals.* Therefore, in this point x_0, $u_{s_1}(x)$ differs from an infinite number of the functions β by more than $\sigma/2$ and consequently in a neighborhood of length δ'' about the point x_0 it differs from them by more than $\sigma/4$.

Let the functions from which $u_{s_1}(x)$ thus differs be

$$(\beta') \qquad u_{t1}, \quad u_{t_2}, \cdots.$$

Of the differences,

$$u_{s_1} - u_{t_1}, \qquad u_{s_1} - u_{t_2}, \cdots$$

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*This obvious fact may be regarded as a special case of a more general theorem. See: *Rendiconti della R. Accad. dei Lincei, ser. 4, vol. 1, p. 262, 1885, or Memorie della R. Accad. delle Scienze. Bologna, ser. 5, vol. 8, p. 130, 1899.*

there will be an infinite number having the same sign, *i. e.*, there will certainly be either an infinite number of functions greater than u_{s_1}, or an infinite number less than u_{s_1}.

Suppose, for example, there are an infinite number greater than u_{s_1}, and denote them by

$$(\beta'') \qquad u_{r_1}, u_{r_2}, \cdots .$$

Starting from u_{r_1} as previously from u_{s_1} we may reason on the β'' as we have already done on the β; and we find that among the β'' there are an infinite number which in an interval δ'' differ from u_{r_1} by more than $\sigma/4$, the differences being all of the same sign. These functions we may denote by

$$(\beta''') \qquad u_{i_1}, u_{i_2}, \cdots .$$

The reasoning may be continued indefinitely in the same way.

But now we notice that between the curve representing u_{s_1} and the set of curves representing functions of the set β'' there lies an area of magnitude $\omega_1 \geqq \delta'' \sigma/4$ bounded by two curves parallel to u_{s_1} and by two ordinates.

Between u_{r_1}, which is one of the β'', and the set β''' there is similarly an area

$$\omega_2 \geqq \delta'' \frac{\sigma}{4};$$

and, moreover, ω_2 is external to ω_1.

This reasoning we may continue indefinitely, obtaining a set of mutually exclusive areas $\omega_1, \omega_2 \cdots$ each greater than $\delta''\sigma/4$. But these areas must lie in a region of finite area. Therefore they cannot be infinite in number. Hence the functions β are finite in number.

There may be several or an infinite number of sets of functions, which, like the functions β, have the property of differing, each from each of the others by more than σ in some point, but the number of functions composing any one set cannot exceed

$$\frac{4(L-l)(b-a)}{\delta''\sigma}.$$

Let one of these sets be

$$(\beta) \qquad u_{s_1}(x), \quad u_{s_2}(x), \cdots, \quad u_{s_p}(x).$$

Of the remaining functions there can be none which differ from these by more than σ in any point, for if there were we should annex them to the set.

It follows that if we consider the neighborhoods

$$\begin{gathered} u_{s_1}(x)-\sigma\cdots u_{s_1}(x)+\sigma, \\ u_{s_2}(x)-\sigma\cdots u_{s_2}(x)+\sigma, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ u_{s_p}(x)-\sigma\cdots u_{s_p}(x)+\sigma, \end{gathered}$$

then in one at least of these neighborhoods there must fall an infinite number of functions belonging to the given set. For if this were not true there would be an infinite number of functions differing from each of the β by more than σ in some point, which is impossible.

We have therefore proved that in the given set x there is a set of functions

$$(a') \qquad u_{m_1}(x), \quad u_{m_2}(x), \cdots,$$

infinite in number, which in every point x differ from each other by less than 2σ. Similarly we may prove that in the set a' there is a sub-set a'' of functions which, taken in pairs, differ in every point x by less than σ^2. In the set a'' we can then find a sub-set a''', the functions in which differ from one another by less than σ^3 in every point. This reasoning may be indefinitely continued.

Let us take now a function $u_{t_1}(x)$ from the set a (for example, u_{t_1} may be the function of lowest index), a function $u_{t_2}(x)$ from the set a', a function $u_{t_3}(x)$ from the set a'', etc. Then if $\sigma < 1$ the set

$$u_{t_1}(x), \quad u_{t_2}(x), \quad u_{t_3}(x), \cdots$$

satisfies the conditions of §VII, and consequently converges to a continuous limiting function, namely, to the function formed by the series

$$u_{t_1}(x)+\Big((u_{t_2}(x)-u_{t_1}(x)\Big)+\Big(u_{t_3}(x)-u_{t_2}(x)\Big)\cdots .$$

The condition is therefore *sufficient* and the theorem is proved.

It is obvious that there may be several or even an infinite number of limiting functions to which converge sets of functions taken from the set a.

4. Consider now, more generally, a set of functions defined by a certain law in the interval $a \cdots b$, all of them being contained between finite limits l and L. Let $u(x)$ be any one of them and denote the set by

$$G \equiv \{u(x)\}.$$

Two functions, $\phi(x)$ and $\psi(x)$ are said to determine a neighborhood of the function $v(x)$ if $\phi(x) < v(x) < \psi(x)$.

If now, however $\phi(x)$ and $\psi(x)$ be taken, provided merely that they satisfy the above inequality, there exist an infinite number of functions of the set G such that $\phi(x) < u(x) < \psi(x)$, then $v(x)$ is said to be a limiting function of the set. It need not however be, itself, a function of the set.

We see at once that, if G contains an equably continuous set of functions, it admits at least one limiting function.

5. **Theorem.*** An infinite set of continuous functions will be equably continuous in the interval $a \cdots b$, if the difference quotient

$$\frac{u(x_1) - u(x_2)}{x_1 - x_2}$$

is, for every $u(x)$ of the set, and for every pair of values x_1 and x_2 possible in the interval $a \cdots b$, comprised between two finite numbers l and L.

For let $|L|$ be the greater of the two numbers $|l|$ and $|L|$. Let h be an arbitrarily taken interval lying in the interval $a \cdots b$; and x_1 and x_2, those points of h which give $u(x)$ its maximum and minimum values respectively.

Then

$$l \leqq \frac{u(x_1) - u(x_2)}{x_1 - x_2} \leqq L.$$

Whence

$$\frac{u(x_1) - u(x_2)}{|x_1 - x_2|} \leqq |L|.$$

Therefore

$$u(x_1) - u(x_2) \leqq |x_1 - x_2| \; |L|,$$

and

$$D_{u,h} \leqq h|L|,$$

$D_{u,h}$ being the oscillation of $u(x)$ in the interval h.

Since this is true from every function $u(x)$ of the set, it follows that, σ being taken arbitrarily small, then in every interval of length

$$h < \frac{\sigma}{|L|}$$

$u(x)$ will oscillate by less than σ. But this is the condition for equable continuity.

Bologna, Italy,
December, 1902.

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*Arzelà, *loc. cit.*, p. 232.